

## STATISTICS

## Paper – I

Time Allowed : Three Hours

Maximum Marks : 200

## Question Paper Specific Instructions

*Please read each of the following instructions carefully before attempting questions :*

*There are **EIGHT** questions in all, out of which **FIVE** are to be attempted.*

*Questions no. 1 and 5 are compulsory. Out of the remaining **SIX** questions, **THREE** are to be attempted selecting at least **ONE** question from each of the two Sections A and B.*

*Attempts of questions shall be counted in sequential order. Unless struck off, attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question-cum-Answer Booklet must be clearly struck off.*

*All questions carry equal marks. The number of marks carried by a question/part is indicated against it.*

*Answers must be written in **ENGLISH** only.*

*Unless otherwise mentioned, symbols and notations have their usual standard meanings.*

*Assume suitable data, if necessary and indicate the same clearly.*

## SECTION A

**Q1.** (a) (i) Show that for arbitrary events  $A_1, A_2, \dots, A_n$ ,

$$P\left(\bigcup_{i=1}^n A_i\right) \leq \sum_{i=1}^n P(A_i).$$

(ii) If  $P_1$  and  $P_2$  are probability measures, prove that

$P = \alpha P_1 + (1 - \alpha) P_2$  is also a probability measure, ( $0 \leq \alpha \leq 1$ ).

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(b) Let  $(X_1, X_2)$  denote quantitative scores on test 1, and  $(Y_1, Y_2)$  be verbal scores on test 2. If  $\text{Cov}(X_1, Y_1) = 5$ ,  $\text{Cov}(X_1, Y_2) = 1$ ,  $\text{Cov}(X_2, Y_1) = 2$  and  $\text{Cov}(X_2, Y_2) = 8$ , compute the covariance between total quantitative score  $(X_1 + X_2)$  and total verbal score  $(Y_1 + Y_2)$ . 8

(c) Derive the lower bound for the variance of an unbiased estimator of  $\sigma^2$  of  $N(\mu, \sigma^2)$ . Hence show that  $s^2 = \frac{1}{n-1} \sum (x_i - \bar{x})^2$  does not attain this bound. Why is it still the best unbiased estimator? 8

(d) Obtain  $100(1 - \alpha)\%$  confidence interval for the difference  $(p_1 - p_2)$  of success probabilities of two independent Bernoulli distributions. 8

(e) Show that the characteristic function (ch. fn.) of a random variable determines its distribution function (d.f.) uniquely. 8

**Q2.** (a) If  $g$  is a continuous function and  $X_n \xrightarrow{P} X$ , then show that  $g(X_n) \xrightarrow{P} g(X)$ . 10

(b) For the Pareto distribution with pdf

$$f(x | \alpha) = \frac{\alpha}{x_0} \left( \frac{x_0}{x_1} \right)^{1+\alpha}, \quad x > 0, \alpha > 0,$$

show that the d.f. is  $\left(1 - \frac{x_0}{x}\right)^\alpha$  and sketch it for  $\alpha = \frac{1}{2}$  and  $\alpha = 2$ . Also show that  $\text{Var}(X)$  does not exist for  $\alpha \leq 2$ . (Sketches to be shown on plain paper). 10

(c) Given a random sample from

$$f(x | \theta) = \theta x^{\theta-1} e^{-x^\theta}, \quad x > 0, \theta > 0.$$

Show that the loglikelihood equation has a unique root and that it provides a strongly consistent estimator. 10

(d) Let  $X$  have a Poisson distribution with mean  $\theta$ . Assume that  $\theta$  has a  $\Gamma(p, \sigma)$  distribution, that is,

$$\pi(\theta) = \frac{\theta^{p-1} \sigma^p e^{-\sigma\theta}}{\Gamma(p)}, \quad \theta > 0, \sigma > 0, p \geq 1.$$

Show that the posterior distribution given  $(x_1, \dots, x_n)$  is

Gamma  $(p + \sum_{i=1}^n x_i, \sigma + n)$ . Hence obtain the Bayes estimator of  $\theta$  with

respect to a squared-error loss. 10

- Q3.** (a) Derive the likelihood ratio test of  $H_0 : \theta \leq \theta_0$  versus  $H_1 : \theta > \theta_0$  given a random sample from

$$f(x|\theta) = \frac{1}{2} \exp\left\{-\frac{1}{2}(x-\theta)\right\}, x > \theta, \theta > 0, \text{ considering } \alpha = 0.05. \quad 10$$

- (b) Consider a random sample  $X_1, X_2, \dots, X_n$  from  $U(0, \theta)$ . Show that the range  $R = X_{(n)} - X_{(1)}$  is an ancillary statistic. 10
- (c) Obtain the level- $\alpha$  UMPU test of  $H_0 : \sigma^2 \leq \sigma_0^2$  against  $H_1 : \sigma^2 > \sigma_0^2$  of  $N(\mu, \sigma^2)$  and its power function ( $\mu$  is unknown). 10
- (d) Using an appropriate pivotal quantity, derive a  $100(1 - \alpha)\%$  confidence interval for  $\mu$  of  $N(\mu, \sigma^2)$ , where  $\sigma^2$  is unknown. Is the C.I. also UMAU? 10

- Q4.** (a) Briefly explain the implementation of SPRT and state its optimal property. Illustrate it for testing  $\theta = \frac{1}{3}$  versus  $\theta = \frac{2}{3}$  for the distribution

$$f(x|\theta) = \theta^{\frac{1-x}{2}} (1-\theta)^{\frac{1+x}{2}}, x = -1, 1. \quad 10$$

- (b) A fair coin is tossed  $n$  times and  $S_n$ , the sum of number of heads, is noted. Use it to find a lower bound for the probability that  $\frac{S_n}{n}$  differs from  $\frac{1}{2}$  by less than 0.1 when  $n = 100$ , and show that  $\frac{S_n}{n} \xrightarrow{P} \frac{1}{2}$  as  $n \rightarrow \infty$ . 10
- (c) Stating the hypothesis, explain the two-sample Wilcoxon-Mann-Whitney test and derive the mean of the test statistic. 10
- (d) Define convergence in distribution. Show that the maximum of a random sample from  $U(0, \theta)$  converges to an exponential distribution. 10

## SECTION B

- Q5.** (a) Given the Gauss-Markov linear model  $(\mathbf{Y}, \mathbf{A}\boldsymbol{\theta}, \sigma^2 \mathbf{I})$  with  $E(Y_1) = \theta_1 + \theta_2 - \theta_3$ ,  
 $E(Y_2) = 2\theta_1 - \theta_2 - \theta_3$ ,  $E(Y_3) = \theta_1 - 2\theta_2$  and  
 $E(Y_4) = 3\theta_1 - 2\theta_3$ . Find a necessary and sufficient condition for  
 $\mathbf{a}'\boldsymbol{\theta} = a_1\theta_1 + a_2\theta_2 + a_3\theta_3$  to be estimable. Hence examine the estimability  
of  $\theta_1 + \theta_2 - 2\theta_3$ . 8

(b) Given  $\mathbf{X} \sim N_3 (\boldsymbol{\mu}' = (2, 4, 3), \boldsymbol{\Sigma} = \begin{bmatrix} 8 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 1 & 3 \end{bmatrix})$ ,

- (i) find the regression function of  $X_1$  on  $X_2$  and  $X_3$ , and  
(ii) compute the conditional variance of  $X_1$  given  $X_2$  and  $X_3$ . 8

- (c) Define connectedness of a block design. Examine if the following design  
is connected : 8

$$B_1 : (T_1, T_2, T_3), B_2 : (T_3, T_4, T_5), B_3 : (T_3, T_4),$$

$$B_4 : (T_1, T_2, T_2), B_5 : (T_4, T_4, T_5).$$

- (d) Describe two-stage sampling with a real life example (stating clearly the  
first and second stage units) and compare it with cluster sampling. 8
- (e) What is the importance of confounding in factorial experiments ? Give a  
layout of a  $2^3$  factorial experiment with 2 replicates so as to confound  
partially the interaction effects ABC and BC. 8

- Q6.** (a) Define multiple and partial correlation coefficients. Suppose that the  
correlation matrix of  $(X_1, X_2, X_3, X_4)'$  is

$$\mathbf{R} = \begin{bmatrix} 1 & 0.6 & -0.36 & -0.53 \\ & 1 & -0.46 & -0.37 \\ & & 1 & 0.37 \\ & & & 1 \end{bmatrix}.$$

Find

- (i) the multiple correlation coefficient between  $X_1$  and the other  
3 variables, and  
(ii) the partial correlation coefficient  $\rho_{X_1 X_2 - X_3 X_4}$ . 10

(b) Suppose that  $Y_i, i = 1, \dots, 6$  is uncorrelated with common variance  $\sigma^2$  and  $E(Y_1) = \beta_1 = E(Y_2), E(Y_3) = \beta_2 = E(Y_4)$  and  $E(Y_5) = \beta_1 + \beta_2 = E(Y_6)$ . Obtain four mutually orthogonal contrasts of  $(Y_i)$  belonging to the error space. Find BLUE of  $\beta_1$  and  $\beta_2$ . 10

(c) If  $\mathbf{Q}$  denotes the vector of adjusted treatment totals in a block design, find (i)  $\mathbf{Q}'\mathbf{J}$  ( $\mathbf{J}$  is a vector of 1's), (ii)  $E(\mathbf{Q})$ , and (iii) covariance matrix of  $\mathbf{Q}$ . 10

(d) Define a regression estimator of population mean. Show that this estimator is more efficient than the sample mean under SRSWOR using large sample approximation. 10

27. (a) Discuss the main effects and interaction effects of a  $2^3$  factorial experiment as orthogonal contrasts. Also describe how to compute the sum of squares due to these contrasts. 10

(b) Explain the concept and utility of principal components. Find the first principal component and its variance of the random vector with covariance matrix  $\Sigma = \begin{bmatrix} 9 & 3 & 3 \\ & 9 & 3 \\ & & 9 \end{bmatrix}$ . 10

(c) Discuss the need for PPS Sampling illustrating with an example from real life. Define Horvitz-Thompson estimator and show that it is unbiased for the population total. 10

(d) Show that a split-plot design in an RBD with  $p$  main plot treatments and  $q$  sub-plot treatments provides more precise comparisons among sub-plot treatments relative to an RBD with  $pq$  plots. 10

28. (a) Suppose that a variable of interest is distributed as  $U(b, b + h), b > 0, h > 0$ . Let the range be divided into  $L$  strata of equal sizes. An SRS of size  $\frac{n}{L}$  is selected from each stratum. Denoting by  $V$  and  $V_1$ , the variances under an SRS of size  $n$ , and a stratified random sample as described above, respectively, show that  $\frac{V_1}{V} = \frac{1}{L^2}$ . Interpret this result. 10

(b) Define multiple linear regression model. Find the unbiased estimators of the parameters of this model. State the sampling distribution of these estimators under the assumption of normality. 10

(c) Explain how one missing value in an RBD is estimated so as to minimise the error sum of squares. Set up the ANOVA table for analysis in this situation. 10

(d) Suppose that a sample of size  $N = 20$  is drawn from  $\mathbf{X} \sim N_3(\boldsymbol{\mu}, \Sigma)$  which yields the sample mean vector and sample covariance matrix as

$$\bar{\mathbf{x}} = (21.05, 21.65, 28.95)'$$

$$S = \begin{pmatrix} 2.2605 & 2.1763 & 1.6342 \\ & 2.6605 & 1.8237 \\ & & 2.4710 \end{pmatrix}.$$

Construct the Likelihood Ratio Test of the hypothesis

$$H_0 : \begin{pmatrix} \mu_2 - \mu_1 \\ \mu_3 - \mu_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

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