MATHEMATICS
Paper – II

Time Allowed : **Three** Hours	Maximum Marks : **200**

Question Paper Specific Instructions

Please read each of the following instructions carefully before attempting questions :

*There are **EIGHT** questions in all, out of which **FIVE** are to be attempted.*

*Questions no. **1** and **5** are **compulsory**. Out of the remaining **SIX** questions, **THREE** are to be attempted selecting at least **ONE** question from each of the two Sections A and B.*

Attempts of questions shall be counted in sequential order. Unless struck off, attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question-cum-Answer Booklet must be clearly struck off.

All questions carry equal marks. The number of marks carried by a question/part is indicated against it.

Unless otherwise mentioned, symbols and notations have their usual standard meanings.

Assume suitable data, if necessary, and indicate the same clearly.

*Answers must be written in **ENGLISH** only.*

SECTION A

Q1. (a) Prove that a subgroup of a cyclic group is cyclic. Let G be a cyclic group with generator a. If the order of G is infinite, then prove that G is isomorphic to $(\mathbb{Z}, +)$. *8*

(b) Find the relative extrema of the function

$f(x, y) = 4y^3 + x^2 - 12y^2 - 36y + 2.$ *8*

(c) Prove that in the interval $0 < x < 1$, the function $f(x) = x^2$ is uniformly continuous while $f(x) = \frac{1}{x}$ is not uniformly continuous. *8*

(d) Prove that $x_1 = 2$, $x_2 = 1$, $x_3 = 0$ is a feasible solution to the following set of equations :

$$2x_1 - x_2 + 3x_3 = 3$$
$$-6x_1 + 3x_2 + 7x_3 = -9$$

Is the solution basic ? Justify your answer. If the solution is not basic, reduce it to a basic feasible one. *8*

(e) Find a bilinear transformation which maps the points $z = 0, -i, -1$ into $w = i, 1, 0$ respectively. *8*

Q2. (a) (i) Prove that every group is isomorphic to a group of permutations. *5*

(ii) Let $A = \{1, 2, 3\}$ and let S_3 denote the symmetric group on 3 elements. Then is S_3 an abelian or non-abelian group ? *5*

(b) (i) Find the volume of the region above the xy-plane bounded by the paraboloid $z = x^2 + y^2$ and the cylinder $x^2 + y^2 = a^2$. *7*

(ii) Prove that $\lim\limits_{M \to \infty} \int_0^M \frac{dx}{x^4 + 4} = \frac{\pi}{8}$. *8*

(c) (i) Let $f(z) = ln\,(1 + z)$. Expand $f(z)$ in a Taylor series about $z = 0$. Determine the region of convergence of the series. 8

(ii) Find Laurent series about the indicated singularity for the function,

$$\frac{e^z}{(z-1)^2};\ z = 1.$$ 7

Q3. (a) (i) Prove that if $u_n(x)$, $n = 1, 2, 3, \ldots\ldots$ are continuous in $[a, b]$ and if $\Sigma\, u_n(x)$ converges uniformly to the sum $S(x)$ in $[a, b]$, then $S(x)$ is continuous in $[a, b]$. 5

(ii) Prove that an absolutely convergent series is convergent. Show that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots.$ is conditionally convergent. 5

(b) (i) If N is a normal subgroup of a group G and if H is any subgroup of G, then prove that

$$H \vee N = HN = NH$$

where $H \vee N$ denotes the join of H and N. 8

(ii) State the Second Isomorphism Theorem of groups and apply it to the case $G = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, $H = \mathbb{Z} \times \mathbb{Z} \times \{0\}$ and $N = \{0\} \times \mathbb{Z} \times \mathbb{Z}$. 7

(c) Consider the LPP :

Minimize

$$z = 10x_1 + 2x_2$$

subject to

$$x_1 + 2x_2 + 2x_3 \geq 1$$

$$x_1 - 2x_3 \geq -1$$

$$x_1 - x_2 + 3x_3 \geq 3,$$

$$x_i \geq 0, \text{ for } i = 1, 2, 3.$$

Solve the dual of the above LPP and find the minimum value of z. 15

Q4. (a) (i) State and prove Cauchy's integral formula. Thus evaluate

$$\oint_C \frac{\cos z}{z - \pi}\, dz,$$

where C is the circle $|z - 1| = 3$. *8*

(ii) State the Residue Theorem and apply it to evaluate

$$\oint_C \frac{e^z\, dz}{(z-1)(z+3)^2}$$

where C is given by $|z| = \frac{3}{2}$. *7*

(b) Prove that the integral domain $\mathbb{Z}$ is a Unique Factorization Domain and a Euclidean Domain. *10*

(c) Five workers perform five jobs and the operating cost is given below, but there is a restriction that the worker C cannot perform the third job and B cannot perform the fifth job. Find the optimal assignment and the optimal assignment cost. *15*

	I	II	III	IV	V
A	24	29	18	32	19
B	17	26	34	22	–
C	27	16	–	17	25
D	22	18	28	30	24
E	28	16	31	24	27

SECTION B

Q5. (a) Consider a particle of mass m moving in a plane under attractive force $\frac{k}{r^2}$ directed towards the origin, where $k > 0$. Using the polar coordinates (r, θ) write the corresponding Lagrangian and obtain the equations of motion. Also show that the angular momentum is conserved. *8*

(b) A function f, defined on [0, 1], is such that $f(0) = 0$, $f\left(\frac{1}{2}\right) = -1$, $f(1) = 0$.

Find the quadratic polynomial p(x) which agrees with f(x) for $x = 0, \frac{1}{2}, 1$.

If $\left|\frac{d^3f}{dx^3}\right| \leq 1$ for $0 \leq x \leq 1$, show that $|f(x) - p(x)| \leq \frac{1}{12}$ for $0 \leq x \leq 1$. *8*

(c) Draw the logic circuit which realises the Boolean function $L = (A + B) . (A + C) + C (A + B . C)$ and simplify it. Draw the simplified circuit also. *8*

(d) In a 2-dimensional flow there are sources at (a, 0), (– a, 0) and sinks at (0, a), (0, –a), all are of equal strength. Determine the stream function and show that the circle through these four points is a streamline. *8*

(e) Solve

$$u_{xx} + \frac{10}{3} u_{xy} + u_{yy} = -\sin(x + y)$$ *8*

Q6. (a) Find the solution of

$$u_x - uu_y + u = 0$$

for the initial values $x_0(s) = 0$, $y_0(s) = s$, $u_0(s) = -2s$.

Does the solution break down for any finite x ? Is the solution unique ? *15*

(b) Find a root of the equation $\sin x + \cos x = 1$, lying in (0, 2), by Regula-Falsi method, correct up to four significant digits. *10*

(c) For a dynamical system having two degrees of freedom, the Lagrangian is given by $L = \frac{m}{2}(a^2 \dot{q}_1^2 + \dot{q}_2^2) - \frac{k}{2}(a^2 + q_2^2)$, where q_1 and q_2 are generalized coordinates.

Find the corresponding Hamiltonian and derive the Hamiltonian equations of motion.

Show further that the generalized momentum corresponding to q_1 is constant.

Show that the system exhibits a simple harmonic motion with respect to the generalized coordinate q_2. *15*

Q7. (a) Solve : *10*

$$u_{tt} - u_{xx} = 0, \quad 0 < x < z, \quad t > 0$$

$$u(0, t) = u(2, t) = 0,$$

$$u(x, 0) = \sin^3 \frac{\pi x}{2},$$

$$u_t(x, 0) = 0.$$

(b) Write down the flow-chart of Runge-Kutta method of 4^{th} order to find $y(0{\cdot}8)$ for $\frac{dy}{dx} = xy$, $y(0) = 2$, taking $h = 0{\cdot}2$.

Also solve the above IVP to find $y(0{\cdot}4)$ by Runge-Kutta method (4^{th} order). *15*

(c) Consider 2-dimensional Navier-Stokes equations of a steady fluid flow.

Show that there exists a stream function $\Psi(x, y)$ for such a flow.

Find the equation satisfied by $\Psi(x, y)$. *15*

Q8. (a) Show that

$$f(x, y, z, p, q) = x^2p^2 + y^2q^2 - 4 = 0$$

and $g(x, y, z, p, q) = qy - a = 0$,

where a is a constant, are compatible and hence solve $f(x, y, z, p, q) = 0$. Is it complete integral ? *15*

(b) State the sufficient condition for convergence of the Gauss-Seidel iteration method and solve the following system of equations by using this method : *15*

$$6{\cdot}7x_1 + 1{\cdot}1x_2 + 2{\cdot}2x_3 = 20{\cdot}5$$

$$2{\cdot}1x_1 - 1{\cdot}5x_2 + 8{\cdot}4x_3 = 28{\cdot}8$$

$$3{\cdot}1x_1 + 9{\cdot}4x_2 - 1{\cdot}5x_3 = 22{\cdot}9$$

(correct up to 3-significant digits)

(c) There is a doublet at (c, 0) in a 2-dimensional flow. A cylinder of radius a (a < c) with z-axis as axis of the cylinder was introduced into the flow. Find the complex potential and image system for the flow. *10*