MATHEMATICS

Paper – II

*Time Allowed : **Three** Hours*

*Maximum Marks : **200***

Question Paper Specific Instructions

Please read each of the following instructions carefully before attempting questions :

*There are **EIGHT** questions in all, out of which **FIVE** are to be attempted.*

*Questions no. **1** and **5** are **compulsory**. Out of the remaining **SIX** questions, **THREE** are to be attempted selecting at least **ONE** question from each of the two Sections A and B.*

Attempts of questions shall be counted in sequential order. Unless struck off, attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question-cum-Answer Booklet must be clearly struck off.

All questions carry equal marks. The number of marks carried by a question/part is indicated against it.

Unless otherwise mentioned, symbols and notations have their usual standard meanings.

Assume suitable data, if necessary, and indicate the same clearly.

*Answers must be written in **ENGLISH** only.*

SECTION A

Q1. (a) Let F be a finite field of characteristic p, where p is a prime. Then show that there is an injective homomorphism from $\mathbb{Z}_p$ (group of integers modulo p) to F. Also show that number of elements in F is p^n, for some positive integer n. *8*

(b) Let $\mathbb{R}$ denote the set of real numbers and Q denote the set of rational numbers. If $x \in \mathbb{R}$, $x > 0$ and $y \in \mathbb{R}$, then show that there exists a positive integer n such that $nx > y$. Use it to show that if $x < y$, then there exists $p \in Q$ such that $x < p < y$. *8*

(c) Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function. Then show that f is Riemann integrable on [a, b]. *8*

(d) Prove that the linear programming problem

Maximize

$$z = 3x_1 + 2x_2$$

subject to the constraints :

$$2x_1 + x_2 \leq 2$$

$$3x_1 + 4x_2 \geq 12$$

$$x_1, x_2 \geq 0$$

does not admit an optimum basic feasible solution. *8*

(e) Compute the integral

$$\oint_C \frac{1 + 2z + z^2}{(z-1)^2 (z+2)} dz$$

where C is $|z| = 3$. *8*

Q2. (a) Find all the Sylow p-subgroups of S_4 and show that none of them is normal. *10*

(b) Suppose $\{f_n\}$ is a sequence of functions defined on $[a, b]$ and $\lim_{n \to \infty} f_n(x) = f(x)$, and $x \in [a, b]$. Put $M_n = \sup_{x \in [a, b]} |f_n(x) - f(x)|$.

Then show that

(i) f_n converges to f uniformly on $[a, b]$ if and only if $M_n \to 0$ as $n \to \infty$. 7

(ii) If $|f_n(x)| \leq M_n$, $(x \in [a, b], n = 1, 2, \ldots)$, then $\sum_{n=1}^{\infty} f_n$ converges uniformly on $[a, b]$ if $\sum_{n=1}^{\infty} M_n$ converges. 8

(c) Find a bilinear transformation $w = f(z)$ which maps the upper half plane $\text{Im}(z) \geq 0$ onto the unit disk $|w| \leq 1$. 15

Q3. (a) (i) Prove that every bounded and monotonically increasing sequence is convergent and converges to lub (least upper bound) of the sequence. 5

(ii) If $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$, $\forall\, n \in \mathbb{N}$, then using Cauchy criterion for convergence of the sequence, show that $\{a_n\}$ is not convergent. 5

(b) (i) Let P be a Sylow p-subgroup of a group G and H is any p-subgroup of G such that HP = PH. Then show that $H \subseteq P$. 7

(ii) Show that every group of order 15 is cyclic. 8

(c) Employ duality to solve the following linear programming problem : 15

Maximize

$$z = 2x_1 + x_2$$

subject to the constraints :

$$x_1 + 2x_2 \leq 10$$

$$x_1 + x_2 \leq 6$$

$$x_1 - x_2 \leq 2$$

$$x_1 - 2x_2 \leq 1$$

$$x_1, x_2 \geq 0$$

Q4. (a) (i) Find an upper bound for the absolute value of the integral $I = \int_C e^z \, dz$, where C is the line segment joining the points (0, 0) and (1, 3). *8*

(ii) Find the length of the curve C defined by

$z(t) = (1 - 2i)t^3, \; -1 \le t \le 1.$ *7*

(b) Prove that R[x] is a principal ideal domain if and only if R is a field. *10*

(c) Find the initial basic feasible solution to the following transportation problem by the North-West corner rule and then optimize it. *15*

	To			Availability
From	7	3	4	2
	2	1	3	3
	3	4	6	5
Demand	4	1	5	10

SECTION B

Q5. (a) Equation of any cone with vertex at the point (a, b, c) is of the form $f\left(\frac{x-a}{z-c}, \frac{y-b}{z-c}\right) = 0$. Find the partial differential equation of the cone. *8*

(b) Given f(1) = 4, f(2) = 5, f(7) = 5 and f(8) = 4. Find the value of f(6) and also the value of x for which f(x) is maximum or minimum. *8*

(c) (i) If x = 0·101010101E0001010 and y = 0·100010110E0000110, then find x – y. *4*

(ii) Draw the map of the Boolean function $F = x'yz + xy'z' + xyz + xyz'$. Also simplify the function. *4*

(d) A rod of length 2a revolves with uniform angular velocity ω about a vertical axis through a smooth joint at one extremity of the rod so that it describes a cone of semi-vertical angle α. Prove that the direction of reaction at the hinge makes with the vertical, an angle $\tan^{-1}\left[\frac{3}{4}\tan\alpha\right]$. *8*

(e) Verify that the equation $yz(y + z)dx + xz(x + z)dy + xy(x + y)dz = 0$ is integrable and find its solution. *8*

Q6. (a) Find the system of equations for obtaining the general equation of surfaces orthogonal to the family given by

$$x(x^2 + y^2 + z^2) = Cy^2,$$

where C is a parameter. *10*

(b) Write down an algorithm for Simpson's $\frac{1}{3}$ rule. Hence, compute $\int_0^1 x^2(1-x)\,dx$ correct up to three decimal places with step size h = 0·1 and compare the result with its exact value. *6+9*

(c) If the velocity of an incompressible fluid at the point (x, y, z) is given by (– Ay, Ax, 0), then prove that the surfaces intersecting the stream lines orthogonally exist and are the planes through z-axis, although the velocity potential does not exist. Discuss the nature of the fluid flow. *15*

Q7. (a) Solve the following system of equations by Gauss-Jordan method : *15*

$$2x + y - 3z = 11$$
$$4x - 2y + 3z = 8$$
$$-2x + 2y - z = -6$$

(b) Verify that $w = i\,k \log\left(\frac{z - ia}{z + ia}\right)$ is the complex potential of a steady fluid flow about a circular cylinder, the plane y = 0 being a rigid boundary.

Further show that the fluid exerts a downward force of magnitude $\left(\frac{\pi \rho k^2}{2a}\right)$ per unit length on the cylinder, where ρ is the fluid density. *15*

(c) Find the solution of the partial differential equation

$$z = \frac{1}{2}(p^2 + q^2) + (p - x)(q - y); \;\; p = \frac{\partial z}{\partial x}, \; q = \frac{\partial z}{\partial y}$$

which passes through the x-axis, using Cauchy's method of characteristics. *10*

Q8. (a) A particle of unit mass is projected so that its total energy is h in a field of force of which the potential energy is φ(r) at a distance r from the origin. By employing the principle of energy and least action, show that the path is given by the following differential equation :

$$c^2\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right] = r^4\,[h - \phi(r)],$$

where c is a constant. *15*

(b) Find the real root of the equation $e^x - 3x = 0$, by Newton–Raphson method, correct up to four decimal places. *10*

(c) Find a complete integral of the partial differential equation

$$(p^2 + q^2)x = pz; \quad p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}$$

using Charpit's method and hence deduce the solution which passes through the curve $x = 0$, $z^2 = 4y$. *15*