Indian Forest Service
(Main) Exam, 2021

ZCVB-U-MTH

# MATHEMATICS

## Paper – I

*Time Allowed : **Three** Hours*

*Maximum Marks : **200***

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### Question Paper Specific Instructions

***Please read each of the following instructions carefully before attempting questions :***

*There are **EIGHT** questions in all, out of which **FIVE** are to be attempted.*

*Questions no. **1** and **5** are **compulsory**. Out of the remaining **SIX** questions, **THREE** are to be attempted selecting at least **ONE** question from each of the two Sections A and B.*

*Attempts of questions shall be counted in sequential order. Unless struck off, attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question-cum-Answer Booklet must be clearly struck off.*

*All questions carry equal marks. The number of marks carried by a question/part is indicated against it.*

*Unless otherwise mentioned, symbols and notations have their usual standard meanings.*

*Assume suitable data, if necessary, and indicate the same clearly.*

*Answers must be written in **ENGLISH** only.*

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## SECTION A

**Q1.** (a) Consider the following quadratic form :

$$q(x, y, z) = 2x^2 + 2y^2 + 6z^2 + 2xy - 6yz - 6zx,$$

where (x, y, z) are the coordinates of the vector X with respect to the standard basis {(1, 0, 0), (0, 1, 0), (0, 0, 1)} of $R^3$. Find the expression of q(x, y, z) with respect to the basis

$$B = \left\{\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\right\}.$$

Is q positive definite ? Justify your answer. *8*

(b) Prove that the product of two Hermitian matrices A, B is Hermitian if and only if A and B commute. Give an example of a pair of 3 × 3 symmetric matrices such that their product is again symmetric (do not consider only diagonal matrices) and also check whether they commute or not. *8*

(c) Using Beta and Gamma functions, evaluate the following integrals : *4+4*

(i) $$\int_0^2 x(8 - x^3)^{1/3}\, dx$$

(ii) $$\int_0^1 \frac{x^2\, dx}{\sqrt{1 - x^5}}$$

(d) Evaluate $$\iint_R x^2\, dx\, dy,$$

where R is the region in the first quadrant bounded by the hyperbola xy = 16 and the lines y = x, y = 0 and x = 8. *8*

(e) Find the equation of the plane passing through the points (1, – 1, 1) and (– 2, 1, – 1) and perpendicular to the plane 2x + y + z + 5 = 0. *8*

**Q2.** (a) Express the polynomial $f(x) = x^2 + 4x - 3$ over R as linear combination of polynomials $e_1 = x^2 - 2x + 5$, $e_2 = 2x^2 - 3x$, $e_3 = x + 3$. Also, show that the set $\{e_1, e_2, e_3\}$ forms a basis of all quadratic polynomials over R. *10*

(b) Find the shortest distance between the line $y = 10 - 2x$ and the ellipse

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

using Lagrange's method of multipliers. *15*

(c) Find the equation of the cone whose vertex is (1, 2, 1) and which passes through the circle $x^2 + y^2 + z^2 = 5$, $x + y - z = 1$. *15*

**Q3.** (a) Does $f(x) = x + \frac{1}{x}$ in $\left[\frac{1}{2}, 3\right]$ satisfy the conditions of the mean value theorem ? If yes, then justify your answer and find $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \left(a = \frac{1}{2}, b = 3\right).$$ *10*

(b) Given the matrix $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$, find a similarity transformation that diagonalises the matrix A. *15*

(c) Show that the straight lines whose direction cosines are given by the equations $al + bm + cn = 0$ and $ul^2 + vm^2 + wn^2 = 0$ (where a, b, c, u, v, w are constants) are parallel if $\frac{a^2}{u} + \frac{b^2}{v} + \frac{c^2}{w} = 0$ and perpendicular if $a^2(v + w) + b^2(w + u) + c^2(u + v) = 0$. *15*

**Q4.** (a) Find the equation of the sphere passing through the points (1, 1, 2), (1, – 1, 2) and having centre on the line $x + y - z - 1 = 0 = 2x + y - z - 2$. *10*

(b) Using the Cayley-Hamilton theorem, find the inverse of the matrix $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & -2 \\ 4 & 2 & 1 \end{bmatrix}$. *15*

(c) Find the whole area included between the curve $x^2 y^2 = a^2 (y^2 - x^2)$ and its asymptotes. *15*

## SECTION B

**Q5.** (a) Solve the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \frac{e^x}{1+e^x}$$

by the method of variation of parameters. *8*

(b) Solve the differential equation

$$y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right).$$ *8*

(c) A particle is projected in a direction making an angle $\alpha$ with the horizon. It passes through the two points $(x_1, y_1)$ and $(x_2, y_2)$. Prove that

$$\tan\alpha = \frac{y_1 R}{Rx_1 - x_1^2} = \frac{x_2^2 y_1 - x_1^2 y_2}{x_1 x_2 (x_2 - x_1)},$$

where R denotes the horizontal range. *8*

(d) Four light rods are joined smoothly to form a quadrilateral ABCD. Let P and Q be the mid-points of an opposite pair of rods and these points are connected by a string in a state of tension T. Let R and S be the mid-points of the other opposite pair of rods and these points are connected by a light rod in a state of thrust X. Show that

$$T \cdot (RS) = X \cdot (PQ).$$ *8*

(e) If $\vec{F} = \left(y\frac{\partial\phi}{\partial z} - z\frac{\partial\phi}{\partial y}\right)\hat{i} + \left(z\frac{\partial\phi}{\partial x} - x\frac{\partial\phi}{\partial z}\right)\hat{j} + \left(x\frac{\partial\phi}{\partial y} - y\frac{\partial\phi}{\partial x}\right)\hat{k}$,

then prove that

$$\vec{F} - (\vec{r} \times \nabla\phi) = \vec{F} \cdot \vec{r} = \vec{F} \cdot \nabla\phi = 0.$$ *8*

**Q6.** (a) Solve the differential equation

$$(D^4 + D^2 + 1)y = e^{-x/2} \cos\left(\frac{1}{2} x \sqrt{3}\right). \quad 10$$

(b) A particle is moving in a medium with central acceleration P. The medium is a resisting medium in which resistance = $kv^2$, v being the velocity.

Let s be the arc-length; $(r, \theta)$ be plane polar coordinates; $u = \frac{1}{r}$ and $M_0$ be the initial moment of momentum about the centre of force. Show that the equation of the path of the particle is

$$Pe^{2ks} = M_0^2 u^2 \left(u + \frac{d^2u}{d\theta^2}\right). \quad 15$$

(c) Let $\vec{a}$ and $\vec{b}$ be any two vector point functions defined on Euclidean space $R^3$. Derive the vector identity for $\nabla(\vec{a} \cdot \vec{b})$. Verify that identity for grad(grad $\phi$ . grad $\psi$), where $\phi = 3x^2y$, $\psi = xz^2 - 2y$. 15

**Q7.** (a) State Gauss' Divergence Theorem completely. Verify the theorem for a field vector $\vec{f} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ taken over the region bounded by the cylinder $x^2 + y^2 = 9$; $z = 0$, $z = 4$. 10

(b) Find the general solution of the differential equation

$$(1 + 2x)^2 \frac{d^2y}{dx^2} - 6(1 + 2x) \frac{dy}{dx} + 16y = 8(1 + 2x)^2. \quad 15$$

(c) Given a solid in the shape of a double cone bounded by two equal circular ends. The solid floats in a liquid, whose density is twice that of the cone, with its axis horizontal. Prove that the equilibrium is stable or unstable according as the semi-vertical angle is less than or greater than 60°. 15

**Q8.** (a) If the mass density at any point of a cord varies as the radius of curvature of the curve in which it hangs freely under gravity, then prove that this curve is the catenary of uniform strength. *10*

(b) (i) Reduce the differential equation $axyp^2 + (x^2 - ay^2 - b)p - xy = 0$, $\left(p = \frac{dy}{dx}\right)$ to Clairaut's form and find the general solution. *8*

(ii) Find the singular solution of the differential equation

$$9p^2(2 - y)^2 = 4(3 - y), \left(p = \frac{dy}{dx}\right).$$ *7*

(c) Prove that :

(i) Principal normals at consecutive points on a curve in a space do not intersect unless its torsion is zero. *7*

(ii) Principal normal of a curve in a space will be binormal of another curve if the curvature of the given curve is proportional to $(k^2 + z^2)$. *8*