

# MATHEMATICS

## Paper - I

Time Allowed : Three Hours

Maximum Marks : 200

### Question Paper Specific Instructions

**Please read each of the following instructions carefully before attempting questions :**

There are **EIGHT** questions in all, out of which **FIVE** are to be attempted.

Questions no. **1** and **5** are compulsory. Out of the remaining **SIX** questions, **THREE** are to be attempted selecting at least **ONE** question from each of the two Sections A and B.

Attempts of questions shall be counted in sequential order. Unless struck off, attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question-cum-Answer Booklet must be clearly struck off.

All questions carry equal marks. The number of marks carried by a question/part is indicated against it.

Answers must be written in **ENGLISH** only.

Unless otherwise mentioned, symbols and notations have their usual standard meanings.

Assume suitable data, if necessary, and indicate the same clearly.

### SECTION A

**Q1.** (a) Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$  be given by

$$T(x, y, z) = (2x - y, 2x + z, x + 2z, x + y + z).$$

Find the matrix of  $T$  with respect to standard basis of  $\mathbb{R}^3$  and  $\mathbb{R}^4$  (i.e.,  $(1, 0, 0)$ ,  $(0, 1, 0)$ , etc.). Examine if  $T$  is a linear map. 8

(b) Show that  $\frac{x}{(1+x)} < \log(1+x) < x$  for  $x > 0$ . 8

(c) Examine if the function  $f(x, y) = \frac{xy}{x^2 + y^2}$ ,  $(x, y) \neq (0, 0)$  and  $f(0, 0) = 0$

is continuous at  $(0, 0)$ . Find  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$  at points other than origin. 8

- (d) If the point (2, 3) is the mid-point of a chord of the parabola  $y^2 = 4x$ , then obtain the equation of the chord. 8

- (e) For the matrix  $A = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ , obtain the eigen value and get the value of  $A^4 + 3A^3 - 9A^2$ . 8

- Q2.** (a) After changing the order of integration of  $\int_0^{\infty} \int_0^{\infty} e^{-xy} \sin nx \, dx \, dy$ , show that  $\int_0^{\infty} \frac{\sin nx}{x} \, dx = \frac{\pi}{2}$ . 10

- (b) A perpendicular is drawn from the centre of ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  to any tangent. Prove that the locus of the foot of the perpendicular is given by  $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$ . 10

- (c) Using mean value theorem, find a point on the curve  $y = \sqrt{x-2}$ , defined on [2, 3], where the tangent is parallel to the chord joining the end points of the curve. 10

- (d) Let T be a linear map such that  $T : V_3 \rightarrow V_2$  defined by  $T(e_1) = 2f_1 - f_2$ ,  $T(e_2) = f_1 + 2f_2$ ,  $T(e_3) = 0f_1 + 0f_2$ , where  $e_1, e_2, e_3$  and  $f_1, f_2$  are standard basis in  $V_3$  and  $V_2$ . Find the matrix of T relative to these basis.

Further take two other basis  $B_1[(1, 1, 0) (1, 0, 1) (0, 1, 1)]$  and  $B_2[(1, 1) (1, -1)]$ . Obtain the matrix  $T_1$  relative to  $B_1$  and  $B_2$ . 10

- Q3.** (a) For the matrix  $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ , find two non-singular matrices P and Q such that  $PAQ = I$ . Hence find  $A^{-1}$ . 10
- (b) Using Lagrange's method of multipliers, find the point on the plane  $2x + 3y + 4z = 5$  which is closest to the point  $(1, 0, 0)$ . 10
- (c) Obtain the area between the curve  $r = 3(\sec \theta + \cos \theta)$  and its asymptote  $x = 3$ . 10
- (d) Obtain the equation of the sphere on which the intersection of the plane  $5x - 2y + 4z + 7 = 0$  with the sphere which has  $(0, 1, 0)$  and  $(3, -5, 2)$  as the end points of its diameter is a great circle. 10
- Q4.** (a) Examine whether the real quadratic form  $4x^2 - y^2 + 2z^2 + 2xy - 2yz - 4xz$  is a positive definite or not. Reduce it to its diagonal form and determine its signature. 10
- (b) Show that the integral  $\int_0^{\infty} e^{-x} x^{\alpha-1} dx$ ,  $\alpha > 0$  exists, by separately taking the cases for  $\alpha \geq 1$  and  $0 < \alpha < 1$ . 10
- (c) Prove that  $\sqrt{2z} = \frac{2^{2z-1}}{\sqrt{\pi}} \sqrt{z} \sqrt{z + \frac{1}{2}}$ . 10
- (d) A plane  $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$  cuts the coordinate plane at A, B, C. Find the equation of the cone with vertex at origin and guiding curve as the circle passing through A, B, C. 10

## SECTION B

- Q5.** (a) Obtain the curve which passes through (1, 2) and has a slope =  $\frac{-2xy}{x^2 + 1}$ .  
Obtain one asymptote to the curve. 8
- (b) Solve the dE to get the particular integral of  $\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = x^2 \cos x$ . 8
- (c) A weight W is hanging with the help of two strings of length  $l$  and  $2l$  in such a way that the other ends A and B of those strings lie on a horizontal line at a distance  $2l$ . Obtain the tension in the two strings. 8
- (d) From a point in a smooth horizontal plane, a particle is projected with velocity  $u$  at angle  $\alpha$  to the horizontal from the foot of a plane, inclined at an angle  $\beta$  with respect to the horizon. Show that it will strike the plane at right angles, if  $\cot \beta = 2 \tan (\alpha - \beta)$ . 8
- (e) If E be the solid bounded by the xy plane and the paraboloid  $z = 4 - x^2 - y^2$ , then evaluate  $\iint_S \bar{F} \cdot dS$  where S is the surface bounding the volume E and  $\bar{F} = (zx \sin yz + x^3) \hat{i} + \cos yz \hat{j} + (3zy^2 - e^{\lambda^2 + y^2}) \hat{k}$ . 8
- Q6.** (a) A stone is thrown vertically with the velocity which would just carry it to a height of 40 m. Two seconds later another stone is projected vertically from the same place with the same velocity. When and where will they meet? 10
- (b) Using the method of variation of parameters, solve  $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y = x^2 e^x$ . 10
- (c) Water is flowing through a pipe of 80 mm diameter under a gauge pressure of 60 kPa, with a mean velocity of 2 m/s. Find the total head, if the pipe is 7 m above the datum line. 10
- (d) Evaluate  $\iint_S (\nabla \times \bar{f}) \cdot \hat{n} dS$  for  $\bar{f} = (2x - y) \hat{i} - yz^2 \hat{j} - y^2z \hat{k}$  where S is the upper half surface of the sphere  $x^2 + y^2 + z^2 = 1$  bounded by its projection on the xy plane. 10

- Q7. (a) State Stokes' theorem. Verify the Stokes' theorem for the function  $\vec{f} = x\hat{i} + z\hat{j} + 2y\hat{k}$ , where  $c$  is the curve obtained by the intersection of the plane  $z = x$  and the cylinder  $x^2 + y^2 = 1$  and  $S$  is the surface inside the intersected one. 15
- (b) A uniform rod of weight  $W$  is resting against an equally rough horizon and a wall, at an angle  $\alpha$  with the wall. At this condition, a horizontal force  $P$  is stopping them from sliding, implemented at the mid-point of the rod. Prove that  $P = W \tan(\alpha - 2\lambda)$ , where  $\lambda$  is the angle of friction. Is there any condition on  $\lambda$  and  $\alpha$ ? 15
- (c) Obtain the singular solution of the differential equation 10
- $$y^2 - 2pxy + p^2(x^2 - 1) = m^2, \quad p = \frac{dy}{dx}.$$

- Q8. (a) A body immersed in a liquid is balanced by a weight  $P$  to which it is attached by a thread passing over a fixed pulley and when half immersed, is balanced in the same manner by weight  $2P$ . Prove that the density of the body and the liquid are in the ratio  $3 : 2$ . 10
- (b) Solve the differential equation 10
- $$\frac{dy}{dx} - y = y^2(\sin x + \cos x).$$
- (c) Prove that  $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$ , if and only if either  $\vec{b} = \vec{0}$  or  $\vec{c}$  is collinear with  $\vec{a}$  or  $\vec{b}$  is perpendicular to both  $\vec{a}$  and  $\vec{c}$ . 10
- (d) A particle is acted on a force parallel to the axis of  $y$  whose acceleration is  $\lambda y$ , initially projected with a velocity  $a\sqrt{\lambda}$  parallel to  $x$ -axis at the point where  $y = a$ . Prove that it will describe a catenary. 10

