IES/ISS EXAM, 2015

0007441

A-LVV-O-UVA

STATISTICS

Paper I

Time Allowed : Three Hours *Maximum Marks : 200*

INSTRUCTIONS

Please read each of the following instructions carefully before attempting the questions :

There are EIGHT questions divided under TWO sections.

Candidate has to attempt FIVE questions in ALL.

Questions no. 1 and 5 are compulsory and out of the remaining, THREE are to be attempted choosing at least ONE from each section.

The number of marks carried by a question/part is indicated against it.

Unless otherwise mentioned, symbols and notations have their usual standard meanings.

Assume suitable data, if necessary, and indicate the same clearly.

Candidates should attempt questions/parts as per the instructions given in the section.

All parts and sub-parts of a question are to be attempted together in the answer book.

Attempts of questions shall be counted in chronological order. Unless struck off, attempt of a question shall be counted even if attempted partly.

Any page or portion of the page left blank in the answer book must be clearly struck off.

Answers must be written in ENGLISH only.

Required tables (Normal Distribution table and 't' table) are attached with the question paper.

SECTION A

1. Answer ***all*** of the following : *5×8=40*

(a) For random variables X, Y, show that

$V[Y] = E_X[V(Y \mid X)] + V_X[E(Y \mid X)]$.

(b) Prove that for $r = 1, 2, \ldots, n$

$$\frac{1}{\Gamma(r)} \int_{\mu}^{\infty} t^{r-1} e^{-t} \, dt = \sum_{x=0}^{r-1} \frac{e^{-\mu} \mu^x}{x!}.$$

(c) Let X have pdf

$$f(x) = \begin{cases} \frac{1}{3}, & -1 < x < 2 \\ 0, & \text{otherwise.} \end{cases}$$

Obtain the cdf of $Y = X^2$.

(d) Let X be a random variable with $E(X) = 3$, $E(X^2) = 13$. Use Chebyshev's inequality to obtain $P(-2 < X < 8)$.

(e) Using Central Limit Theorem, show that

$$e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} \doteq \frac{1}{2}.$$

2. (a) Let C be a circle of unit area with centre at origin and let S be a square of unit area with $\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, -\frac{1}{2}\right), \left(-\frac{1}{2}, \frac{1}{2}\right)$ and $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ as the four vertices. If X and Y be two independent standard normal variates, show that

$$\iint_C \phi(x)\, \phi(y)\, dx\, dy \geq \iint_S \phi(x)\, \phi(y)\, dx\, dy$$

where $\phi(\cdot)$ is the pdf of $N(0, 1)$ distribution.

(b) Let X follow log-normal with parameters μ and σ^2. Find the distribution of

$Y = aX^b,\ a > 0,\ -\infty < b < \infty.$

(c) Let $\{X_n\}$ be a sequence of pairwise, uncorrelated random variables with

$E(X_i) = \mu_i$ and $V(X_i) = \sigma_i^2,\ i = 1, 2, \ldots$

If $\sum_{i=1}^{n} \sigma_i^2 \to \infty$ as $n \to \infty$, then show that

$$\sum_{i=1}^{n} \left(\frac{X_i - \mu_i}{\sum_{j=1}^{n} \sigma_j^2} \right) \xrightarrow{p} 0 \text{ as } n \to \infty.$$ 20

3. (a) Let $X_1, X_2, \ldots, X_n$ be independent Poisson variates with $E(X_i) = \mu_i$. Find the conditional distribution of $X_1, \ldots, X_n \,\Big|\, \sum_{i=1}^{n} X_i = y$.

(b) Let Y_1 denote the first order statistic in a random sample of size n from a distribution that has the pdf $f(x) = \begin{cases} e^{-(x-\theta)}, & \theta < x < \infty \\ 0, & \text{otherwise} \end{cases}$

Obtain the distribution of $Z_n = n(Y_1 - \theta)$.

(c) Two points are chosen at random on a line of unit length. Find the probability that each of the 3 line segments will have length greater than $\frac{1}{4}$. 40

4. (a) Obtain the characteristic function of X whose pdf is

$$f(x) = \frac{\lambda}{\pi}, \frac{1}{\lambda^2 + (x-\mu)^2}, -\infty < x < \infty.$$

(b) Let $\{X_n\}$ be a sequence of random variables with

$$P(X_n = \pm n^{\alpha}) = \frac{1}{2} n^{-\alpha}$$

$$P(X_n = 0) = 1 - n^{-\alpha}$$

For what values of α does weak law of large numbers (WLLN) hold ?

(c) Three urns U_1, U_2 and U_3 each contain 5 black balls and 7 white balls initially. A ball is drawn at random from U_1 and 2 balls of the drawn colour are added to U_2. Then a ball is drawn at random from U_2 and 3 balls of the drawn colour are added to U_3. Find the probability of drawing a white ball from U_3. 40

SECTION B

5. Answer ***all*** of the following : *5×8=40*

(a) Let (X, Y) be distributed as bivariate normal BVN $(3, 1; 13, 25; \frac{3}{5})$. Calculate $P(4 < Y < 11 \cdot 84 \mid X = 7)$.

(b) With 3 variables X_1, X_2 and X_3, it is given that $r_{13} = 0 \cdot 71$, $R_{1 \cdot 23} = 0 \cdot 78$. Find $r_{12 \cdot 3}$.

(c) Suppose the given values of x_i are such that $a \leq x_i \leq b$ for $i = 1, 2, ..., n$. Show that

$$0 \leq s^2 \leq \frac{(b-a)^2}{4}.$$

(d) Let X have F(m, n) distribution. Obtain $E(X^{-r})$, $r > 0$.

(e) If X follows binomial $b(n_1, p_1)$ distribution and Y follows $b(n_2, p_2)$, provide an appropriate exact test at level α for $H_0 : p_1 = p_2$ against $H_1 : p_1 > p_2$.

6. (a) By using Euler – Maclaurin formula, find the sum $\frac{1}{51^2} + \frac{1}{53^2} + \ldots\ldots\ldots + \frac{1}{99^2}$.

(b) Given the random samples

X : 1, 5, 7, 9, 15, 17, 21, 23

Y : 2, 6, 10, 12, 18, 20, 26, 28, 32

from the populations having the distribution function respectively as F_1 and F_2, test the hypothesis

$$H_0 : F_1 = F_2$$

$$\text{against} \quad H_1 : F_1 \neq F_2$$

at 5% level of significance by the Wald – Wolfowitz run test. It is given that the critical number of runs at sample sizes (8, 9) at 5% level is 5.

(c) Compute Yule's coefficient of association (Q) and Yule's coefficient of colligation (Y) for the following table :

		Disease on-set	
		Yes	No
Medicine used	A	19	587
	B	193	2741

40

7. (a) By making use of the difference table and a suitable interpolation formula, find the number of students who obtained less than 45 marks in an examination, from the following table :

Marks	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80
Number of Students	31	42	51	35	31

(b) Consider the two samples as follows :

Sample I = 6, 7, 8, 10, 12, 14, 16, 23

Sample II = 9, 11, 13, 15, 17, 18, 19, 20, 24

Test whether the samples have come from the same populations using Wilcoxon-Mann-Whitney (WMW) test at 10% level of significance. [You can use normal approximation].

(c) For 20 pairs of heights of father (X) and son (Y) measured in cm, the following data were obtained :

$$\bar{x} = 168{\cdot}17, \ \Sigma (x_i - \bar{x})^2 = 777{\cdot}80$$

$$\Sigma (y_i - \bar{y})^2 = 939{\cdot}42, \ y = 9{\cdot}25 + 0{\cdot}932x$$

Test whether the cut on the X-axis can be assumed to be zero, at 5% level of significance. *40*

8. (a) Compute the value of $\int_4^{5\cdot2} ln\, x\, dx$ by Simpson's $\frac{1}{3}^{rd}$ rule. Given that $ln\ 4\cdot0 = 1\cdot39$, $ln\ 4\cdot2 = 1\cdot43$, $ln\ 4\cdot4 = 1\cdot48$, $ln\ 4\cdot6 = 1\cdot53$, $ln\ 4\cdot8 = 1\cdot57$, $ln\ 5\cdot0 = 1\cdot61$, $ln\ 5\cdot2 = 1\cdot65$.

(b) Let $X_1, X_2, ..., X_{12}$ be a random sample from a normal $N(\mu_1, \sigma^2)$ distribution and $Y_1, Y_2, ..., Y_0$ be another random sample from normal $N(\mu_2, \sigma^2)$, independently of each other. Carry out an appropriate test for testing

$$H_0 : \mu_1 = \mu_2$$

against $H_1 : \mu_1 \neq \mu_2$

at 5% level of significance. It is given that $\overline{x} = 2$, $s_x^2 = 16$, $\overline{y} = 8$ and $s_y^2 = 15$.

(c) Fit the exponential curve $y = a + bx$ to the following data :

x :	0	2	4
y :	5·01	10	31·62

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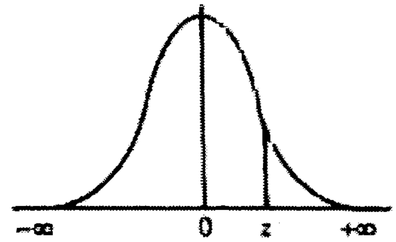
Normal Distribution Table



	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

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t Table

cum. prob	$t_{.50}$	$t_{.75}$	$t_{.80}$	$t_{.85}$	$t_{.90}$	$t_{.95}$	$t_{.975}$	$t_{.99}$	$t_{.995}$	$t_{.999}$	$t_{.9995}$
one-tail	**0.50**	**0.25**	**0.20**	**0.15**	**0.10**	**0.05**	**0.025**	**0.01**	**0.005**	**0.001**	**0.0005**
two-tails	**1.00**	**0.50**	**0.40**	**0.30**	**0.20**	**0.10**	**0.05**	**0.02**	**0.01**	**0.002**	**0.001**
df											
1	0.000	1.000	1.376	1.963	3.078	6.314	12.71	31.82	63.66	318.31	636.62
2	0.000	0.816	1.061	1.386	1.886	2.920	4.303	6.965	9.925	22.327	31.599
3	0.000	0.765	0.978	1.250	1.638	2.353	3.182	4.541	5.841	10.215	12.924
4	0.000	0.741	0.941	1.190	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	0.000	0.727	0.920	1.156	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	0.000	0.718	0.906	1.134	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	0.000	0.711	0.896	1.119	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	0.000	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	0.000	0.703	0.883	1.100	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	0.000	0.700	0.879	1.093	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	0.000	0.697	0.876	1.088	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	0.000	0.695	0.873	1.083	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	0.000	0.694	0.870	1.079	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	0.000	0.692	0.868	1.076	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	0.000	0.691	0.866	1.074	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	0.000	0.690	0.865	1.071	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	0.000	0.689	0.863	1.069	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	0.000	0.688	0.862	1.067	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	0.000	0.688	0.861	1.066	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	0.000	0.687	0.860	1.064	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	0.000	0.686	0.859	1.063	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	0.000	0.686	0.858	1.061	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	0.000	0.685	0.858	1.060	1.319	1.714	2.069	2.500	2.807	3.485	3.768
24	0.000	0.685	0.857	1.059	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	0.000	0.684	0.856	1.058	1.316	1.708	2.060	2.485	2.787	3.450	3.725
26	0.000	0.684	0.856	1.058	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	0.000	0.684	0.855	1.057	1.314	1.703	2.052	2.473	2.771	3.421	3.690
28	0.000	0.683	0.855	1.056	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	0.000	0.683	0.854	1.055	1.311	1.699	2.045	2.462	2.756	3.396	3.659
30	0.000	0.683	0.854	1.055	1.310	1.697	2.042	2.457	2.750	3.385	3.646
40	0.000	0.681	0.851	1.050	1.303	1.684	2.021	2.423	2.704	3.307	3.551
60	0.000	0.679	0.848	1.045	1.296	1.671	2.000	2.390	2.660	3.232	3.460
80	0.000	0.678	0.846	1.043	1.292	1.664	1.990	2.374	2.639	3.195	3.416
100	0.000	0.677	0.845	1.042	1.290	1.660	1.984	2.364	2.626	3.174	3.390
1000	0.000	0.675	0.842	1.037	1.282	1.646	1.962	2.330	2.581	3.098	3.300
z	0.000	0.674	0.842	1.036	1.282	1.645	1.960	2.326	2.576	3.090	3.291
	0%	50%	60%	70%	80%	90%	95%	98%	99%	99.8%	99.9%
	Confidence Level										