General Aptitude (GA)

Q.1 – Q.5 Carry ONE mark Each

Q.1	If '→' denotes increasing order of intensity, then the meaning of the words [drizzle → rain → downpour] is analogous to [_______ → quarrel → feud]. Which one of the given options is appropriate to fill the blank?
(A)	bicker
(B)	bog
(C)	dither
(D)	dodge

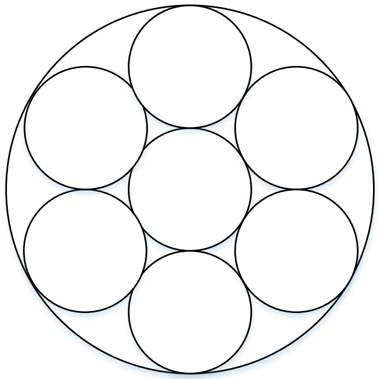
Q.2	Statements: 1. All heroes are winners. 2. All winners are lucky people. Inferences: I. All lucky people are heroes. II. Some lucky people are heroes. III. Some winners are heroes. Which of the above inferences can be logically deduced from statements 1 and 2?
(A)	Only I and II
(B)	Only II and III
(C)	Only I and III
(D)	Only III

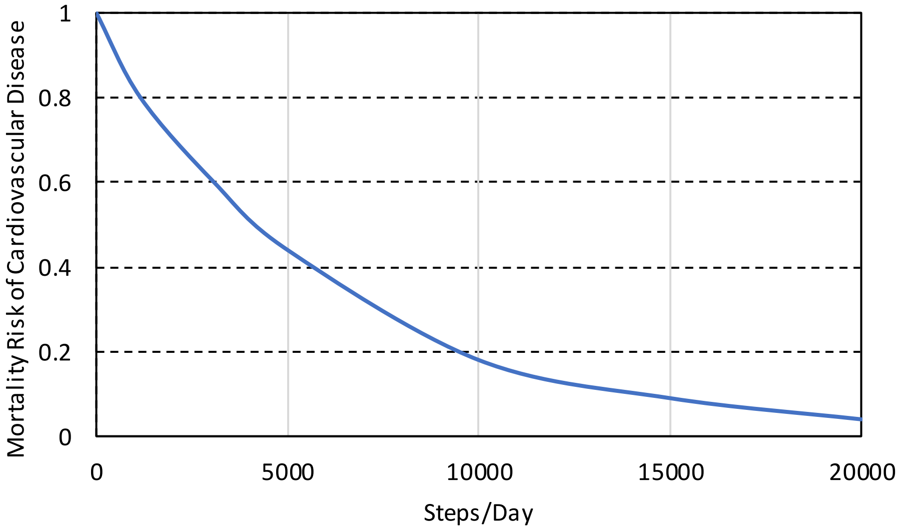
Q.3	A student was supposed to **multiply** a positive real number p with another positive real number q. Instead, the student **divided** p by q. If the percentage error in the student's answer is 80%, the value of q is
(A)	5
(B)	$\sqrt{2}$
(C)	2
(D)	$\sqrt{5}$
Q.4	If the sum of the first 20 consecutive positive odd numbers is divided by 20^2, the result is
(A)	1
(B)	20
(C)	2
(D)	1/2

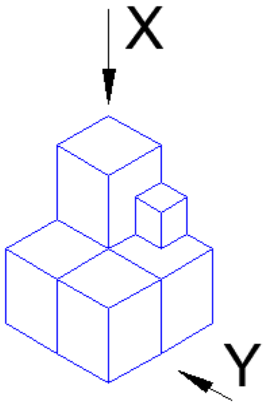
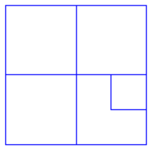
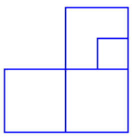
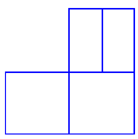
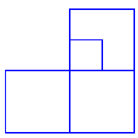
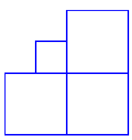
Q.5	The ratio of the number of girls to boys in class VIII is the same as the ratio of the number of boys to girls in class IX. The total number of students (boys and girls) in classes VIII and IX is 450 and 360, respectively. If the number of girls in classes VIII and IX is the same, then the number of girls in each class is
(A)	150
(B)	200
(C)	250
(D)	175

Q.6 – Q.10 Carry TWO marks Each

Q.6	In the given text, the blanks are numbered (i)−(iv). Select the best match for all the blanks. Yoko Roi stands ___(i)___ as an author for standing ___(ii)___ as an honorary fellow, after she stood ___(iii)___ her writings that stand ___(iv)___ the freedom of speech.
(A)	(i) out (ii) down (iii) in (iv) for
(B)	(i) down (ii) out (iii) by (iv) in
(C)	(i) down (ii) out (iii) for (iv) in
(D)	(i) out (ii) down (iii) by (iv) for

Q.7	Seven identical cylindrical chalk-sticks are fitted tightly in a cylindrical container. The figure below shows the arrangement of the chalk-sticks inside the cylinder.  The length of the container is equal to the length of the chalk-sticks. The ratio of the occupied space to the empty space of the container is
(A)	5/2
(B)	7/2
(C)	9/2
(D)	3

Q.8	The plot below shows the relationship between the mortality risk of cardiovascular disease and the number of steps a person walks per day. Based on the data, which one of the following options is true? 
(A)	The risk reduction on increasing the steps/day from 0 to 10000 is less than the risk reduction on increasing the steps/day from 10000 to 20000.
(B)	The risk reduction on increasing the steps/day from 0 to 5000 is less than the risk reduction on increasing the steps/day from 15000 to 20000.
(C)	For any 5000 increment in steps/day the largest risk reduction occurs on going from 0 to 5000.
(D)	For any 5000 increment in steps/day the largest risk reduction occurs on going from 15000 to 20000.

Q.9	Five cubes of identical size and another smaller cube are assembled as shown in Figure A. If viewed from direction X, the planar image of the assembly appears as Figure B.  Figure A  Figure B If viewed from direction Y, the planar image of the assembly (Figure A) will appear as
(A)	
(B)	
(C)	
(D)	

Q.10	Visualize a cube that is held with one of the four body diagonals aligned to the vertical axis. Rotate the cube about this axis such that its view remains unchanged. The magnitude of the minimum angle of rotation is
(A)	120°
(B)	60°
(C)	90°
(D)	180°

Useful data

$\mathbb{Z}$ - the set of integers.

$\mathbb{R}$ - the set of real numbers.

$\mathbb{C}$ - the set of complex numbers.

$\mathrm{Im}(z)$ - imaginary part of the complex number z.

$\mathrm{Re}(z)$ - real part of the complex number z.

$\sup(A)$ - supremum of the set $A \subseteq \mathbb{R}$.

$\inf(A)$ - infimum of the set $A \subseteq \mathbb{R}$.

$M_n(\mathbb{F})$ - the set of $n \times n$ matrices over a field $\mathbb{F}$.

$GL_n(\mathbb{F})$ - the set of $n \times n$ invertible matrices over a field $\mathbb{F}$.

$\mathbb{F}^n$ - the n-dimensional vector space over a field $\mathbb{F}$.

I_n - the $n \times n$ identity matrix.

S_n - the symmetric group on n elements.

$A \setminus B = \{a \in A : a \notin B\}$.

$\mathbb{Z}/n\mathbb{Z}$ - the cyclic group of order n.

$\ln$ - the natural logarithm.

f', f'', f''' - the first-order, second-order, and third-order derivative, respectively, of a one variable function f

MCQ - 1 Mark

11. Consider the following condition on a function $f : \mathbb{C} \to \mathbb{C}$

$$|f(z)| = 1 \quad \text{for all } z \in \mathbb{C} \text{ such that } \operatorname{Im}(z) = 0. \qquad \textbf{(P)}$$

Which one of the following is correct?

(A) There is a non-constant analytic polynomial f satisfying (**P**)

(B) Every entire function f satisfying (**P**) is a constant function

(C) Every entire function f satisfying (**P**) has no zeroes in $\mathbb{C}$

(D) There is an entire function f satisfying (**P**) with infinitely many zeroes in $\mathbb{C}$

12. Let C be the ellipse $\{z \in \mathbb{C} : |z-2| + |z+2| = 8\}$ traversed counter-clockwise. The value of the contour integral

$$\oint_C \frac{z^2 dz}{z^2 - 2z + 2}$$

is equal to

(A) 0

(B) $2\pi i$

(C) $4\pi i$

(D) $-\pi i$

13. Let X be a topological space and $A \subseteq X$. Given a subset S of X, let int(S), ∂S, and $\overline{S}$ denote the interior, boundary, and closure, respectively, of the set S. Which one of the following is NOT necessarily true?

(A) $\mathrm{int}(X \setminus A) \subseteq X \setminus \overline{A}$

(B) $A \subseteq \overline{A}$

(C) $\partial A \subseteq \partial\left(\mathrm{int}(A)\right)$

(D) $\partial\big(\overline{A}\big) \subseteq \partial A$

14. Consider the following limit

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^\infty e^{-x/\varepsilon} \left(\cos(3x) + x^2 + \sqrt{x+4}\right) dx.$$

Which one of the following is correct?

(A) The limit does not exist

(B) The limit exists and is equal to 0

(C) The limit exists and is equal to 3

(D) The limit exists and is equal to π

15. Let $\mathbb{R}[X^2, X^3]$ be the subring of $\mathbb{R}[X]$ generated by X^2 and X^3. Consider the following statements.

I. The ring $\mathbb{R}[X^2, X^3]$ is a unique factorization domain.

II. The ring $\mathbb{R}[X^2, X^3]$ is a principal ideal domain.

Which one of the following is correct?

(A) Both I and II are TRUE

(B) I is TRUE and II is FALSE

(C) I is FALSE and II is TRUE

(D) Both I and II are FALSE

16. Given a prime number p, let $n_p(G)$ denote the number of p–Sylow subgroups of a finite group G. Which one of the following is TRUE for every group G of order 2024?

(A) $n_{11}(G) = 1$ and $n_{23}(G) = 11$

(B) $n_{11}(G) \in \{1, 23\}$ and $n_{23}(G) = 1$

(C) $n_{11}(G) = 23$ and $n_{23}(G) \in \{1, 88\}$

(D) $n_{11}(G) = 23$ and $n_{23}(G) = 11$

17. Consider the following statements.

I. Every compact Hausdorff space is normal.

II. Every metric space is normal.

Which one of the following is correct?

(A) Both I and II are TRUE

(B) I is TRUE and II is FALSE

(C) I is FALSE and II is TRUE

(D) Both I and II are FALSE

18. Consider the topology on $\mathbb{Z}$ with basis $\{S(a,b) : a,b \in \mathbb{Z}$ and $a \neq 0\}$, where

$$S(a,b) = \{an + b : n \in \mathbb{Z}\}.$$

Consider the following statements.

I. $S(a,b)$ is both open and closed for each $a,b \in \mathbb{Z}$ with $a \neq 0$.

II. The only connected set containing $x \in \mathbb{Z}$ is $\{x\}$.

Which one of the following is correct?

(A) Both I and II are TRUE

(B) I is TRUE and II is FALSE

(C) I is FALSE and II is TRUE

(D) Both I and II are FALSE

19. Let $A = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ and $T : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be the linear transformation given by $T(B) = AB$. The characteristic polynomial of T is

(A) $X^4 - 8X^2 + 16$

(B) $X^2 - 4$

(C) $X^2 - 2$

(D) $X^4 - 16$

20. Let $A \in M_n(\mathbb{C})$ be a normal matrix. Consider the following statements.

I. If all the eigenvalues of A are real, then A is Hermitian.

II. If all the eigenvalues of A have absolute value 1, then A is unitary.

Which one of the following is correct?

(A) Both I and II are TRUE

(B) I is TRUE and II is FALSE

(C) I is FALSE and II is TRUE

(D) Both I and II are FALSE

21. Let

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ -1 & 1 & 2 \end{pmatrix}$$

and $\mathbf{b}$ be a 3×1 real column vector. Consider the statements.

I. The Jacobi iteration method for the system $(A + \varepsilon I_3)\mathbf{x} = \mathbf{b}$ converges for any initial approximation and $\varepsilon > 0$.

II. The Gauss–Seidel iteration method for the system $(A + \varepsilon I_3)\mathbf{x} = \mathbf{b}$ converges for any initial approximation and $\varepsilon > 0$.

Which one of the following is correct?

(A) Both I and II are TRUE

(B) I is TRUE and II is FALSE

(C) I is FALSE and II is TRUE

(D) Both I and II are FALSE

22. For the initial value problem

$$y' = f(x, y), \qquad y(x_0) = y_0,$$

generate approximations y_n to $y(x_n)$, $x_n = x_0 + nh$, for a fixed $h > 0$ and $n = 1, 2, 3, \ldots$, using the recursion formula

$$y_n = y_{n-1} + ak_1 + bk_2, \text{ where}$$

$$k_1 = hf(x_{n-1}, y_{n-1}) \text{ and } k_2 = hf(x_{n-1} + \alpha h, y_{n-1} + \beta k_1).$$

Which one of the following choices of a, b, α, β for the above recursion formula gives the Runge–Kutta method of order 2 ?

(A) $a = 1, b = 1, \alpha = 0.5, \beta = 0.5$

(B) $a = 0.5, b = 0.5, \alpha = 2, \beta = 2$

(C) $a = 0.25, b = 0.75, \alpha = 2/3, \beta = 2/3$

(D) $a = 0.5, b = 0.5, \alpha = 1, \beta = 2$

23. Let $u = u(x,t)$ be the solution of

$$\frac{\partial u}{\partial t} - 4\frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < 1, \quad t > 0,$$
$$u(0,t) = u(1,t) = 0, \quad t \geq 0,$$
$$u(x,0) = \sin(\pi x), \quad 0 \leq x \leq 1.$$

Define $g(t) = \int_0^1 (u(x,t))^2 dx$, for $t > 0$. Which one of the following is correct?

(A) g is decreasing on $(0,\infty)$ and $\lim_{t\to\infty} g(t) = 0$

(B) g is decreasing on $(0,\infty)$ and $\lim_{t\to\infty} g(t) = 1/4$

(C) g is increasing on $(0,\infty)$ and $\lim_{t\to\infty} g(t)$ does not exist

(D) g is increasing on $(0,\infty)$ and $\lim_{t\to\infty} g(t) = 3$

24. If y_1 and y_2 are two different solutions of the ordinary differential equation

$$y' + \sin(e^x)y = \cos(e^{x+1}), \quad 0 \leq x \leq 1,$$

then which one of the following is its general solution on $[0,1]$?

(A) $c_1y_1 + c_2y_2, \quad c_1, c_2 \in \mathbb{R}$

(B) $y_1 + c(y_1 - y_2), \quad c \in \mathbb{R}$

(C) $cy_1 + (y_1 - y_2), \quad c \in \mathbb{R}$

(D) $c_1(y_1 + y_2) + c_2(y_1 - y_2), \quad c_1, c_2 \in \mathbb{R}$

25. Consider the following Linear Programming Problem **P**

$$\begin{aligned} \text{minimize} \quad & 5x_1 + 2x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \leq 2, \\ & x_1 + x_2 \geq 1, \\ & x_1, x_2 \geq 0. \end{aligned}$$

The optimal value of the problem **P** is equal to

(A) 5

(B) 0

(C) 4

(D) 2

NAT - 1 Mark

26. Let $p = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right) \in \mathbb{R}^4$ and $f : \mathbb{R}^4 \to \mathbb{R}$ be a differentiable function such that $f(p) = 6$ and $f(\lambda x) = \lambda^3 f(x)$, for every $\lambda \in (0, \infty)$ and $x \in \mathbb{R}^4$. The value of

$$12\frac{\partial f}{\partial x_1}(p) + 6\frac{\partial f}{\partial x_2}(p) + 4\frac{\partial f}{\partial x_3}(p) + 3\frac{\partial f}{\partial x_4}(p)$$

is equal to ________ (answer in integer)

27. The number of non-isomorphic finite groups with exactly 3 conjugacy classes is equal to ________ (answer in integer)

28. Let $f(x, y) = (x^2 - y^2, 2xy)$, where $x > 0, y > 0$. Let g be the inverse of f in a neighborhood of $f(2, 1)$. Then the determinant of the Jacobian matrix of g at $f(2, 1)$ is equal to ________ (round off to TWO decimal places)

29. Let $\mathbb{F}_3$ be the field with exactly 3 elements. The number of elements in $GL_2(\mathbb{F}_3)$ is equal to ________ (answer in integer)

30. Given a real subspace W of $\mathbb{R}^4$, let $W^\perp$ denote its orthogonal complement with respect to the standard inner product on $\mathbb{R}^4$. Let $W_1 = \text{Span}\{(1, 0, 0, -1)\}$ and $W_2 = \text{Span}\{(2, 1, 0, -1)\}$ be real subspaces of $\mathbb{R}^4$. The dimension of $W_1^\perp \cap W_2^\perp$ over $\mathbb{R}$ is equal to ________ (answer in integer)

31. The number of group homomorphisms from $\mathbb{Z}/4\mathbb{Z}$ to S_4 is equal to ________ (answer in integer)

32. Let $a \in \mathbb{R}$ and h be a positive real number. For any twice-differentiable function $f : \mathbb{R} \to \mathbb{R}$, let $P_f(x)$ be the interpolating polynomial of degree at most two that interpolates f at the points $a-h, a, a+h$. Define d to be the largest integer such that any polynomial g of degree d satisfies

$$g''(a) = P_g''(a).$$

The value of d is equal to ________ (answer in integer)

33. Let $P_f(x)$ be the interpolating polynomial of degree at most two that interpolates the function $f(x) = x^2|x|$ at the points $x = -1, 0, 1$. Then

$$\sup_{x\in[-1,1]} |f(x) - P_f(x)| = \_\_\_\_\_\_\_\_$$

(round off to TWO decimal places)

34. The maximum of the function $f(x, y, z) = xyz$ subject to the constraints

$$xy + yz + zx = 12,\ x > 0, y > 0, z > 0,$$

is equal to ________ (round off to TWO decimal places)

35. If the outward flux of $\mathbf{F}(x, y, z) = \left(x^3, y^3, z^3\right)$ through the unit sphere $x^2+y^2+z^2 = 1$ is $\alpha\pi$, then α is equal to ________ (round off to TWO decimal places)

MCQ - 2 Mark

36. Let $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Then

$$\sup\left\{|\,f'(0)| : f \text{ is an analytic function from } \mathbb{D} \text{ to } \mathbb{H} \text{ and } f(0) = \frac{i}{2}\right\}$$

is equal to

(A) $\frac{1}{4}$

(B) $\frac{1}{2}$

(C) 1

(D) 100

37. Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. For which one of the following functions f does there exist a sequence of polynomials in z that uniformly converges to f on S^1?

(A) $f(z) = \overline{z}$

(B) $f(z) = \mathrm{Re}(z)$

(C) $f(z) = e^z$

(D) $f(z) = |z+1|^2$

38. Let $f : [0, 1] \to \mathbb{R}$ be a function. Which one of the following is a sufficient condition for f to be Lebesgue measurable?

(A) $|f|$ is a Lebesgue measurable function

(B) There exist continuous functions $g, h : [0, 1] \to \mathbb{R}$ such that $g \leq f \leq h$ on $[0, 1]$

(C) f is continuous almost everywhere on $[0, 1]$

(D) For each $c \in \mathbb{R}$, the set $\{x \in [0, 1] : f(x) = c\}$ is Lebesgue measurable

39. Let $g : M_2(\mathbb{R}) \to \mathbb{R}$ be given by $g(A) = \mathrm{Trace}(A^2)$. Let $\mathbf{0}$ be the 2×2 zero matrix. The space $M_2(\mathbb{R})$ may be identified with $\mathbb{R}^4$ in the usual manner. Which one of the following is correct?

(A) $\mathbf{0}$ is a point of local minimum of g

(B) $\mathbf{0}$ is a point of local maximum of g

(C) $\mathbf{0}$ is a saddle point of g

(D) $\mathbf{0}$ is not a critical point of g

40. Consider the following statements.

I. There exists a proper subgroup G of $(\mathbb{Q}, +)$ such that $\mathbb{Q}/G$ is a finite group.

II. There exists a subgroup G of $(\mathbb{Q}, +)$ such that $\mathbb{Q}/G$ is isomorphic to $(\mathbb{Z}, +)$.

Which one of the following is correct?

(A) Both I and II are TRUE

(B) I is TRUE and II is FALSE

(C) I is FALSE and II is TRUE

(D) Both I and II are FALSE

41. Let X be the space $\mathbb{R}/\mathbb{Z}$ with the quotient topology induced from the usual topology on $\mathbb{R}$. Consider the following statements.

I. X is compact.

II. $X \setminus \{x\}$ is connected for any $x \in X$.

Which one of the following is correct?

(A) Both I and II are TRUE

(B) I is TRUE and II is FALSE

(C) I is FALSE and II is TRUE

(D) Both I and II are FALSE

42. Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{R}^7$. Let $\Sigma = \{v_1, \ldots, v_5\} \subseteq \mathbb{R}^7$ be a set of unit vectors such that $\langle v_i, v_j \rangle$ is a *non-positive* integer for all $1 \leq i \neq j \leq 5$. Define $N(\Sigma)$ to be the number of pairs (r, s), $1 \leq r, s \leq 5$, such that $\langle v_r, v_s \rangle \neq 0$. The maximum possible value of $N(\Sigma)$ is equal to

(A) 9

(B) 10

(C) 14

(D) 5

43. Let $f(x) = |x| + |x-1| + |x-2|$, $x \in [-1, 2]$. Which one of the following numerical integration rules gives the exact value of the integral

$$\int_{-1}^{2} f(x)dx$$

(A) The Simpson's rule

(B) The trapezoidal rule

(C) The composite Simpson's rule by dividing $[-1, 2]$ into 4 equal subintervals

(D) The composite trapezoidal rule by dividing $[-1, 2]$ into 3 equal subintervals

44. Consider the initial value problem (IVP)

$$y' = e^{-y^2} + 1,$$
$$y(0) = 0.$$

I. IVP has a unique solution on $\mathbb{R}$.

II. Every solution of IVP is bounded on its maximal interval of existence.

Which one of the following is correct?

(A) Both I and II are TRUE

(B) I is TRUE and II is FALSE

(C) I is FALSE and II is TRUE

(D) Both I and II are FALSE

45. Let A be a 2×2 non-diagonalizable real matrix with a real eigenvalue λ and $\mathbf{v}$ be an eigenvector of A corresponding to λ. Which one of the following is the general solution of the system $\mathbf{y}' = A\mathbf{y}$ of first-order linear differential equations?

(A) $c_1 e^{\lambda t}\mathbf{v} + c_2 t e^{\lambda t}\mathbf{v}$, where $c_1, c_2 \in \mathbb{R}$

(B) $c_1 e^{\lambda t}\mathbf{v} + c_2 t^2 e^{\lambda t}\mathbf{v}$, where $c_1, c_2 \in \mathbb{R}$

(C) $c_1 e^{\lambda t}\mathbf{v} + c_2 e^{\lambda t}(t\mathbf{v} + \mathbf{u})$, where $c_1, c_2 \in \mathbb{R}$ and $\mathbf{u}$ is a 2×1 real column vector such that $(A - \lambda I_2)\mathbf{u} = \mathbf{v}$

(D) $c_1 e^{\lambda t}\mathbf{v} + c_2 t e^{\lambda t}(\mathbf{v} + \mathbf{u})$, where $c_1, c_2 \in \mathbb{R}$ and $\mathbf{u}$ is a 2×1 real column vector such that $(A - \lambda I_2)\mathbf{u} = \mathbf{v}$

46. Let $D = \{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } y > 0\}$. If the following second-order linear partial differential equation

$$y^2 \frac{\partial^2 u}{\partial x^2} - x^2 \frac{\partial^2 u}{\partial y^2} + y \frac{\partial u}{\partial y} = 0 \text{ on } D$$

is transformed to

$$\left(\frac{\partial^2 u}{\partial \eta^2} - \frac{\partial^2 u}{\partial \xi^2}\right) + \left(\frac{\partial u}{\partial \eta} + \frac{\partial u}{\partial \xi}\right) \frac{1}{2\eta} + \left(a \frac{\partial u}{\partial \eta} + b \frac{\partial u}{\partial \xi}\right) \frac{1}{2\xi} = 0 \text{ on } D$$

for some $a, b \in \mathbb{R}$, via the co-ordinate transform $\eta = \frac{x^2}{2}$ and $\xi = \frac{y^2}{2}$, then which one of the following is correct?

(A) $a = 2, b = 0$

(B) $a = 0, b = -1$

(C) $a = 1, b = -1$

(D) $a = 1, b = 0$

47. Let $\ell^p = \left\{ x = (x_n)_{n\geq 1} : x_n \in \mathbb{R}, \|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} < \infty \right\}$ for $p = 1, 2$. Let

$$\mathcal{C}_{00} = \{(x_n)_{n\geq 1} : x_n = 0 \text{ for all but finitely many } n \geq 1\}.$$

For $x = (x_n)_{n\geq 1} \in \mathcal{C}_{00}$, define $f(x) = \sum_{n=1}^{\infty} \frac{x_n}{\sqrt{n}}$. Consider the following statements.

I. There exists a continuous linear functional F on $(\ell^1, \|\cdot\|_1)$ such that $F = f$ on $\mathcal{C}_{00}$.

II. There exists a continuous linear functional G on $(\ell^2, \|\cdot\|_2)$ such that $G = f$ on $\mathcal{C}_{00}$.

Which one of the following is correct?

(A) Both I and II are TRUE

(B) I is TRUE and II is FALSE

(C) I is FALSE and II is TRUE

(D) Both I and II are FALSE

MSQ - 2 Mark

48. Let $\ell^2_{\mathbb{Z}} = \left\{ (x_j)_{j\in\mathbb{Z}} : x_j \in \mathbb{R} \text{ and } \sum_{j=-\infty}^{\infty} x_j^2 < \infty \right\}$ endowed with the inner product

$$\langle x, y\rangle = \sum_{j=-\infty}^{\infty} x_j y_j, \qquad x = (x_j)_{j\in\mathbb{Z}},\ y = (y_j)_{j\in\mathbb{Z}} \in \ell^2_{\mathbb{Z}}.$$

Let $T : \ell^2_{\mathbb{Z}} \to \ell^2_{\mathbb{Z}}$ be given by $T((x_j)_{j\in\mathbb{Z}}) = (y_j)_{j\in\mathbb{Z}}$, where

$$y_j = \frac{x_j + x_{-j}}{2}, \quad j \in \mathbb{Z}.$$

Which of the following is/are correct?

(A) T is a compact operator

(B) The operator norm of T is 1

(C) T is a self-adjoint operator

(D) $\mathrm{Range}(T)$ is closed

49. Let X be the normed space $(\mathbb{R}^2, \|\cdot\|)$, where

$$\|(x, y)\| = |x| + |y|, \quad (x, y) \in \mathbb{R}^2.$$

Let $S = \{(x, 0) : x \in \mathbb{R}\}$ and $f : S \to \mathbb{R}$ be given by $f((x, 0)) = 2x$ for all $x \in \mathbb{R}$. Recall that a Hahn–Banach extension of f to X is a continuous linear functional F on X such that $F|_S = f$ and $\|F\| = \|f\|$, where $\|F\|$ and $\|f\|$ are the norms of F and f on X and S, respectively. Which of the following is/are true?

(A) $F(x, y) = 2x + 3y$ is a Hahn–Banach extension of f to X

(B) $F(x, y) = 2x + y$ is a Hahn–Banach extension of f to X

(C) f admits infinitely many Hahn–Banach extensions to X

(D) f admits exactly two distinct Hahn–Banach extensions to X

50. Let $\{[a, b) : a, b \in \mathbb{R}, a < b\}$ be a basis for a topology τ on $\mathbb{R}$. Which of the following is/are correct?

(A) Every (a, b) with $a < b$ is an open set in $(\mathbb{R}, \tau)$

(B) Every $[a, b]$ with $a < b$ is a compact set in $(\mathbb{R}, \tau)$

(C) $(\mathbb{R}, \tau)$ is a first-countable space

(D) $(\mathbb{R}, \tau)$ is a second-countable space

51. Let $T, S : \mathbb{R}^4 \to \mathbb{R}^4$ be two non–zero, non–identity $\mathbb{R}$-linear transformations. Assume $T^2 = T$. Which of the following is/are TRUE?

(A) T is necessarily invertible

(B) T and S are similar if $S^2 = S$ and $\mathrm{Rank}(T) = \mathrm{Rank}(S)$

(C) T and S are similar if S has only 0 and 1 as eigenvalues

(D) T is necessarily diagonalizable

52. Let $p_1 < p_2$ be the two fixed points of the function $g(x) = e^x - 2$, where $x \in \mathbb{R}$. For $x_0 \in \mathbb{R}$, let the sequence $(x_n)_{n\geq 1}$ be generated by the fixed point iteration

$$x_n = g(x_{n-1}), \quad n \geq 1.$$

Which one of the following is/are correct?

(A) $(x_n)_{n\geq 0}$ converges to p_1 for any $x_0 \in (p_1, p_2)$

(B) $(x_n)_{n\geq 0}$ converges to p_2 for any $x_0 \in (p_1, p_2)$

(C) $(x_n)_{n\geq 0}$ converges to p_2 for any $x_0 > p_2$

(D) $(x_n)_{n\geq 0}$ converges to p_1 for any $x_0 < p_1$

53. Which of the following is/are eigenvalue(s) of the Sturm–Liouville problem

$$\begin{aligned} y'' + \lambda y = 0, &\qquad 0 \le x \le \pi, \\ y(0) = y'(0), & \\ y(\pi) = y'(\pi)? & \end{aligned}$$

(A) $\lambda = 1$

(B) $\lambda = 2$

(C) $\lambda = 3$

(D) $\lambda = 4$

54. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that

$$f(x,y) = \begin{cases} \left(1 - \cos \dfrac{x^2}{y^2}\right) \sqrt{x^2+y^2}, & \text{if } y \neq 0, x \in \mathbb{R}, \\ 0, & \text{otherwise.} \end{cases}$$

Which of the following is/are correct?

(A) f is continuous at $(0,0)$, but not differentiable at $(0,0)$

(B) f is differentiable at $(0,0)$

(C) All the directional derivatives of f at $(0,0)$ exist and they are equal to zero

(D) Both the partial derivatives of f at $(0,0)$ exist and they are equal to zero

55. For an integer n, let $f_n(x) = xe^{-nx}$, where $x \in [0,1]$. Let $S := \{f_n : n \geq 1\}$. Consider the metric space $(\mathcal{C}([0,1]), d)$, where

$$d(f,g) = \sup_{x\in[0,1]} \{|f(x) - g(x)|\}, \quad f, g \in \mathcal{C}([0,1]).$$

Which of the following statement(s) is/are true?

(A) S is an equi-continuous family of continuous functions

(B) S is closed in $(\mathcal{C}([0,1]), d)$

(C) S is bounded in $(\mathcal{C}([0,1]), d)$

(D) S is compact in $(\mathcal{C}([0,1]), d)$

56. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be an $\mathbb{R}$-linear transformation such that 1 and 2 are the only eigenvalues of T. Suppose the dimensions of $\mathrm{Kernel}(T - I_4)$ and $\mathrm{Range}(T - 2I_4)$ are 1 and 2, respectively. Which of the following is/are possible (upper triangular) Jordan canonical form(s) of T?

(A)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

(B)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

(C)

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

(D)

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

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57. Let $L^2([-1,1])$ denote the space of all real-valued Lebesgue square-integrable functions on $[-1,1]$, with the usual norm $\|\cdot\|$. Let $\mathcal{P}_1$ be the subspace of $L^2([-1,1])$ consisting of all the polynomials of degree at most 1. Let $f \in L^2([-1,1])$ be such that $\|f\|^2 = \dfrac{18}{5}$, $\int_{-1}^{1} f(x)dx = 2$, and $\int_{-1}^{1} xf(x)dx = 0$. Then

$$\inf_{g \in \mathcal{P}_1} \|f-g\|^2 = \_\_\_\_\_\_\_\_$$

(round off to TWO decimal places)

58. The maximum value of $f(x,y,z) = 10x + 6y - 8z$ subject to the constraints

$$\begin{aligned} 5x - 2y + 6z &\leq 20, \\ 10x + 4y - 6z &\leq 30, \\ x, y, z &\geq 0, \end{aligned}$$

is equal to ________ (round off to TWO decimal places)

59. Let $K \subseteq \mathbb{C}$ be the field extension of $\mathbb{Q}$ obtained by adjoining all the roots of the polynomial equation $(X^2-2)(X^2-3) = 0$. The number of distinct fields F such that $\mathbb{Q} \subseteq F \subseteq K$ is equal to ________ (answer in integer)

60. Let H be the subset of S_3 consisting of all $\sigma \in S_3$ such that

$$\text{Trace}(A_1 A_2 A_3) = \text{Trace}(A_{\sigma(1)} A_{\sigma(2)} A_{\sigma(3)}),$$

for all $A_1, A_2, A_3 \in M_2(\mathbb{C})$. The number of elements in H is equal to ________ (answer in integer)

61. Let $\mathbf{r} : [0,1] \to \mathbb{R}^2$ be a continuously differentiable path from $(0,2)$ to $(3,0)$ and let $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\mathbf{F}(x,y) = (1-2y, 1-2x)$. The line integral of $\mathbf{F}$ along $\mathbf{r}$

$$\int \mathbf{F} \cdot d\mathbf{r}$$

is equal to ________ (round off to TWO decimal places)

62. Let $u = u(x,t)$ be the solution of the initial value problem

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \quad x \in \mathbb{R}, t > 0,$$

$$u(x,0) = 0, \quad x \in \mathbb{R},$$

$$\frac{\partial u}{\partial t}(x,0) = \begin{cases} x^4(1-x)^4, & \text{if } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

If $\alpha = \inf\{t > 0 : u(2,t) > 0\}$, then α is equal to ________ (round off to TWO decimal places)

63. The boundary value problem

$$x^2 y'' - 2xy' + 2y = 0, \qquad 1 \le x \le 2,$$

$$y(1) - y'(1) = 1,$$

$$y(2) - ky'(2) = 4,$$

has infinitely many distinct solutions when k is equal to ________ (round off to TWO decimal places)

64. The global maximum of $f(x,y) = (x^2 + y^2)e^{2-x-y}$ on $\{(x,y) \in \mathbb{R}^2 : x \ge 0, y \ge 0\}$ is equal to ________ (round off to TWO decimal places)

65. Let $k \in \mathbb{R}$ and $D = \{(r, \theta) : 0 < r < 2, 0 < \theta < \pi\}$. Let $u(r, \theta)$ be the solution of the following boundary value problem

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \quad (r, \theta) \in D,$$
$$u(r, 0) = u(r, \pi) = 0, \quad 0 \leq r \leq 2,$$
$$u(2, \theta) = k \sin(2\theta), \quad 0 < \theta < \pi.$$

If $u\left(1, \frac{\pi}{4}\right) = 2$, then the value of k is equal to ________ (round off to TWO decimal places)